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## The Interacting $W_3$ String

H. Lu, C.N. Pope,<sup>\*</sup> S. Schrans<sup>◇</sup> and X.J. Wang

*Center for Theoretical Physics, Texas A&M University,  
College Station, TX 77843-4242, USA.*

### ABSTRACT

We present a procedure for computing gauge-invariant scattering amplitudes in the  $W_3$  string, and use it to calculate three-point and four-point functions. We show that non-vanishing scattering amplitudes necessarily involve external physical states with excitations of ghosts as well as matter fields. The crossing properties of the four-point functions are studied, and it is shown that the duality of the Virasoro string amplitudes generalises in the  $W_3$  string, with different sets of intermediate states being exchanged in different channels. We also exhibit a relation between the scattering amplitudes of the  $W_3$  string and the fusion rules of the Ising model.

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<sup>◇</sup> Onderzoeker I.I.K.W.; On leave of absence from the Instituut voor Theoretische Fysica, K.U. Leuven, Belgium. Address after March 1, 1993: Koninklijke/Shell-Laboratorium, Amsterdam (Shell Research B.V.), Badhuisweg 3, 1031 CM Amsterdam, The Netherlands.

## 1. Introduction

$W$  strings are a natural generalisation of the ordinary bosonic string [1–6]. The spectrum of physical states of a  $W$ -string theory is given by the non-trivial cohomology of its BRST operator. Since the BRST operator is known explicitly only for the case of  $W_3$  [7], most attention has focussed on this example. Considerable progress has been made in determining the spectrum of physical states of the  $W_3$  string [2,4,5,8,9,10] but until now, very little progress has been made in introducing interactions in the theory. The basic reason for this is that there seems to be no way of constructing non-vanishing gauge-invariant correlation functions involving only physical states of standard ghost structure. In this paper we show how such correlation functions can in fact be built. The new ingredient that allows this is the existence, and recent discovery [9,10], of new physical states with non-standard ghost structure, which have excitations of the ghosts as well as of the matter fields. Non-vanishing scattering amplitudes necessarily require that some external physical states have this form. Thus the  $W_3$ -string theory is rather unusual in that such states seem to be essential in constructing an interacting theory.

For the 26-dimensional bosonic string, the physical states all have the form  $|\psi\rangle = |X\rangle \otimes |\text{gh}\rangle$ , where  $|X\rangle$  is built exclusively from creation operators of the 26 matter fields  $X^\mu$  acting on a momentum eigenstate, and  $|\text{gh}\rangle$  is the standard ghost vacuum. For the multi-scalar  $W_3$  string analogous physical states also exist, and have been fully classified [4,5,8]. However, it has recently been appreciated that this does not exhaust the non-trivial cohomology of the BRST operator for the  $W_3$  string [10]. In fact there are many additional physical states, which have a non-standard structure in that they involve excitations of the ghost as well as the matter fields. The occurrence of these states can be attributed to the fact that there does not exist a physical gauge for the  $W_3$  string. For example, there are massive vector states in the  $W_3$  string, indicating the existence of less spacetime gauge symmetry than for the ordinary bosonic string. The phenomenon of physical states with non-standard ghost structure in fact occurs in the bosonic string, but only in the special case when there are two spacetime dimensions [11,12]. However, physical states with non-standard ghost structure occur in the *multi-scalar*  $W_3$  string [10], where the on-shell momenta take *continuous* values, by contrast to the two-scalar bosonic string where the momenta are necessarily discrete.

The purpose of this paper is to develop a procedure for computing gauge-invariant scattering amplitudes for the  $W_3$  string, and to discuss their significance. In section 2, we present a short review of the multi-scalar  $W_3$  string and the construction of its physical states. The construction is exhaustive for physical states with the standard ghost structure. The spectrum of non-standard physical states has not yet been obtained completely, but sufficiently many examples are now known to enable us to investigate some of the features of the interacting theory. The explicit forms of the relevant physical states are relegated to an appendix.

In section 3, we present our procedure for writing down gauge-invariant scattering amplitudes for the  $W_3$  string. We shall illustrate the procedure by computing specific examples of three-point and four-point tachyon scattering amplitudes. In section 4 we discuss the interpretation of these amplitudes, including their crossing and duality properties and their relation to the Ising model. The paper ends with conclusions and comments in section 5.

## 2. Physical states of the multi-scalar $W_3$ string

The key ingredient for determining the physical spectrum of the  $W_3$  string is the construction of the BRST operator [7], which is given by

$$Q_B = \oint dz \left[ c \left( T + \frac{1}{2} T_{\text{gh}} \right) + \gamma \left( W + \frac{1}{2} W_{\text{gh}} \right) \right], \quad (2.1)$$

and is nilpotent provided that the matter currents  $T$  and  $W$  generate the  $W_3$  algebra with central charge  $c = 100$ , and that the ghost currents are chosen to be

$$T_{\text{gh}} = -2b \partial c - \partial b c - 3\beta \partial \gamma - 2\partial \beta \gamma, \quad (2.2)$$

$$\begin{aligned} W_{\text{gh}} = & -\partial \beta c - 3\beta \partial c - \frac{8}{261} [\partial(b \gamma T) + b \partial \gamma T] \\ & + \frac{25}{1566} \left( 2\gamma \partial^3 b + 9\partial \gamma \partial^2 b + 15\partial^2 \gamma \partial b + 10\partial^3 \gamma b \right), \end{aligned} \quad (2.3)$$

where the ghost-antighost pairs  $(c, b)$  and  $(\gamma, \beta)$  correspond respectively to the  $T$  and  $W$  generators. A matter realisation of  $W_3$  with central charge 100 can be given in terms of  $n \geq 2$  scalar fields, as follows [13]:

$$\begin{aligned} T &= -\frac{1}{2}(\partial \varphi)^2 - Q \partial^2 \varphi + T^{\text{eff}}, \\ W &= -\frac{2i}{\sqrt{261}} \left[ \frac{1}{3}(\partial \varphi)^3 + Q \partial \varphi \partial^2 \varphi + \frac{1}{3} Q^2 \partial^3 \varphi + 2\partial \varphi T^{\text{eff}} + Q \partial T^{\text{eff}} \right], \end{aligned} \quad (2.4)$$

where  $Q^2 = \frac{49}{8}$  and  $T^{\text{eff}}$  is an energy-momentum tensor with central charge  $\frac{51}{2}$  that commutes with  $\varphi$ . Since  $T^{\text{eff}}$  has a fractional central charge, it cannot be realised simply by taking free scalar fields. We can however use  $d$  scalar fields  $X^\mu$  with a background-charge vector  $a_\mu$ :

$$T^{\text{eff}} = -\frac{1}{2} \partial X_\mu \partial X^\mu - i a_\mu \partial^2 X^\mu, \quad (2.5)$$

with  $a_\mu$  chosen so that  $\frac{51}{2} = d - 12a_\mu a^\mu$  [4].

Physical states are by definition states that are annihilated by the BRST operator (2.1) but that are not BRST trivial. We shall first consider such states with standard ghost structure, *i.e.*

$$|\chi\rangle = |\varphi, X\rangle \otimes |--\rangle. \quad (2.6)$$

Here  $|--\rangle$  is the standard ghost vacuum, given by

$$|--\rangle = c_1 \gamma_1 \gamma_2 |0\rangle. \quad (2.7)$$

The  $SL(2, C)$  vacuum satisfies

$$c_n |0\rangle = 0, \quad n \geq 2; \quad b_n |0\rangle = 0, \quad n \geq -1, \quad (2.8a)$$

$$\gamma_n |0\rangle = 0, \quad n \geq 3; \quad \beta_n |0\rangle = 0, \quad n \geq -2. \quad (2.8b)$$

The anti-ghost fields  $b, \beta$  have ghost number  $G = -1$ , and the ghost fields  $c, \gamma$  have ghost number  $G = 1$ .\*

For standard states of the form (2.6), the condition of BRST invariance becomes [7]:

$$\begin{aligned} (L_0 - 4)|\varphi, X\rangle &= 0, \\ W_0|\varphi, X\rangle &= 0, \\ L_n|\varphi, X\rangle = W_n|\varphi, X\rangle &= 0, \quad n \geq 1. \end{aligned} \quad (2.9)$$

The consequences of these physical-state conditions have been studied in detail in various papers [2,4,5,8]. The main features that emerge are the following. The excited states can be divided into two kinds, namely those for which there are no excitations in the  $\varphi$  direction, and those where  $\varphi$  is excited too. The latter states are all null, as has been discussed in [5,8], and thus need not be considered further. For the former, we may write  $|\varphi, X\rangle$  as

$$|\varphi, X\rangle = e^{\beta\varphi(0)} |\text{phys}\rangle_{\text{eff}}, \quad (2.10)$$

where  $|\text{phys}\rangle_{\text{eff}}$  involves only the  $X^\mu$  fields and not  $\varphi$ . The physical-state conditions (2.9) now imply that

$$(\beta + Q)(\beta + \frac{6}{7}Q)(\beta + \frac{8}{7}Q) = 0, \quad (2.11)$$

together with the effective physical-state conditions [2,4,5]:

$$\begin{aligned} (L_0^{\text{eff}} - \Delta)|\text{phys}\rangle_{\text{eff}} &= 0, \\ L_n^{\text{eff}}|\text{phys}\rangle_{\text{eff}} &= 0, \quad n \geq 1. \end{aligned} \quad (2.12)$$

The value of the effective intercept  $\Delta$  is 1 when  $\beta = -\frac{6}{7}Q$  or  $-\frac{8}{7}Q$ , and it equals  $\frac{15}{16}$  when  $\beta = -Q$ . Thus these states of the  $W_3$  string are described by two effective Virasoro-string spectra, for an effective energy-momentum tensor  $T^{\text{eff}}$  with central charge  $c = \frac{51}{2}$  and intercepts  $\Delta = 1$  and  $\Delta = \frac{15}{16}$ . The first of these gives a mass spectrum similar to an ordinary

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\* For states, we adopt the convention that the ghost vacuum  $|--\rangle$  has ghost number  $G = 0$ , which means that the  $SL(2, C)$  vacuum  $|0\rangle$  has ghost number  $G = -3$ . This implies that physical states of ghost number  $G$  are obtained by acting on  $|0\rangle$  with operators of ghost number  $(G + 3)$ .

string, with a massless vector at level 1, whilst the second gives a spectrum of purely massive states [5].

As we have indicated in the introduction, the  $W_3$ -string spectrum is much richer than simply that of the standard physical states we have discussed so far. Although the classification of physical states with non-standard ghost structure is as yet incomplete, some classes of such states have been found [10,14]. They contain excitations of the ghost and antighost fields as well as the matter fields. The level number  $\ell$  of these states is defined with respect to the ghost vacuum  $|--\rangle$  given in (2.7). Thus, for example, the  $SL(2, C)$  vacuum  $|0\rangle$  has level number  $\ell = 4$  and ghost number  $G = -3$ , since it can be written as  $\beta_{-2}\beta_{-1}b_{-1}|--\rangle$ . It is straightforward to see that at level  $\ell$ , the allowed ghost numbers of *states* (not necessarily physical) lie in the interval

$$[1 - \sqrt{4\ell + 1}] \leq G \leq [1 + \sqrt{4\ell + 1}], \quad (2.13)$$

where  $[a]$  denotes the integer part of  $a$ . In fact for all known examples of *physical* states [10], the ghost number lies in the restricted interval  $-3 \leq G \leq 5$ .

All the physical states in the  $W_3$  string occur in multiplets [10], each of which may be viewed as being built from what we shall call a *prime state*. We shall discuss the prime states first. All the known examples of prime states occur either at ghost number  $G = 0$  (in the case of states with standard ghost structure), or at  $G = -1$  or  $G = -3$  (in the case of states with non-standard ghost structure). The corresponding operators have ghost numbers  $G = 3$ ,  $G = 2$  or  $G = 0$ . From (2.13), we see that the last of these can arise only when the level number  $\ell$  is  $\geq 4$ , and thus we first discuss the  $G = -1$  prime states. (In fact the examples that we shall be concerned with in this paper involve level numbers  $\leq 3$ .)

The first non-standard physical states arise at level  $\ell = 1$ . They are given by operators of the form

$$V = \left( c\gamma \mp \frac{i}{3\sqrt{58}} \partial\gamma\gamma \right) e^{\beta\varphi + ip \cdot X} \quad (2.14)$$

acting on the  $SL(2, C)$  vacuum  $|0\rangle$  [10]. Just as in the case of the physical states with standard ghost structure, here the corresponding physical states, and indeed all the non-standard physical states we shall consider in this paper, admit a natural interpretation as string-like states for the effective spacetime described by the coordinates  $X^\mu$ . The operator  $V$  in (2.14) describes a physical state if  $\beta = -\frac{4}{7}Q$  and the  $-$  sign is chosen, or if  $\beta = -\frac{3}{7}Q$  and the  $+$  sign is chosen. The former operator leads to a state with effective spacetime intercept  $L_0^{\text{eff}} = \Delta = \frac{1}{2}$ , whilst the latter corresponds to  $\Delta = \frac{15}{16}$ . We denote these physical operators by  $\mathbf{V}_\Delta^G[\beta, p] = \mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p]$  and  $\mathbf{V}_{15/16}^2[-\frac{3}{7}Q, p]$ , and shall adopt similar conventions for the other physical states. The spectrum of the  $G = -1$  level 1 prime states is completed by two discrete states, with  $\beta$  momenta  $-\frac{6}{7}Q$  and  $-\frac{8}{7}Q$ , and spacetime momentum  $p_\mu = 0$  in each case [10].

At level  $\ell = 2$ , there is just one physical state with non-standard ghost structure at  $G = -1$ . It has momentum  $\beta = -\frac{2}{7}Q$  in the  $\varphi$  direction, and effective spacetime intercept  $\Delta = \frac{1}{2}$ . Following the same notation as above, we denote the corresponding  $G = 2$  operator by  $\mathbf{V}_{1/2}^2[-\frac{2}{7}Q, p]$ . Its detailed form is given in (A.11).

At level  $\ell = 3$ , there are three  $G = -1$  physical states with non-standard ghost structure. There is a state with  $\beta = 0$  and  $\Delta = 1$ ; we denote the corresponding operator by  $\mathbf{V}_1^2[0, p]$ . Its detailed form is given in (A.12). Since in this paper we shall not need the two other prime states that occur at this level, we shall not give their explicit form, but merely note that they have  $\beta = -\frac{4}{7}Q$  and  $\beta = -\frac{3}{7}Q$ . A more detailed discussion of physical states with non-standard ghost structure will be given in [15].

Prime states with  $G = -1$  presumably exist at all higher level numbers too, but as yet no classification of them exists. There are also further prime states at  $G = -3$ , [10], which, in view of (2.13), can occur only when the level number satisfies  $\ell \geq 4$ . In fact the first example, at  $\ell = 4$ , is the  $SL(2, C)$  vacuum  $|0\rangle = \beta_{-2}\beta_{-1}b_{-1}|--\rangle$ . This is manifestly BRST invariant and BRST non-trivial, and thus provides a zero-momentum discrete state.

Higher-level  $G = -3$  physical states were first found in [10], in the context of the two-scalar  $W_3$  string, where all physical states have discrete momenta. Some physical states in the two-scalar  $W_3$  string generalise to continuous-momentum physical states in the multi-scalar  $W_3$  string, some generalise to discrete-momentum physical states, and some do not admit generalisations at all. We have seen examples above of states in the first two categories. At level  $\ell = 3$  there are further discrete states in the two-scalar  $W_3$  string that do not generalise to the multi-scalar case. In [10], two  $\ell = 6$  discrete states at  $G = -3$  were found for the two-scalar  $W_3$  string; these are analogues of the ground-ring generators [12] for the discrete states of the two-scalar bosonic string. One of these  $\ell = 6$  states generalises to a discrete state in the multi-scalar  $W_3$  string; it corresponds to a  $G = 0$  operator, with  $\beta = \frac{2}{7}Q$  and spacetime momentum  $p_\mu = 0$ . In [16], four further discrete states in the two-scalar  $W_3$  string were found, at  $\ell = 8$  and  $G = -3$ . Together with the  $\ell = 6$  states described above, they constitute the complete set of “ring generators” for the two-scalar  $W_3$  string [16]. Two of them (a pair with conjugate values of momentum in the second direction) generalise to give a continuous-momentum physical state in the multi-scalar  $W_3$  string, corresponding to a  $G = 0$  operator with  $\beta = \frac{4}{7}Q$  and continuous on-shell spacetime momentum. The detailed expressions for the  $\ell = 6$  and  $\ell = 8$  operators that we have described here are quite complicated. Since we shall not be making use of them in the present paper, we shall not give their explicit form. Details may be found in [10,16].

As we mentioned above, the physical states with non-standard ghost structure arise in multiplets [10], and so far we have described just the *prime states*. For each multiplet, this is the state from which all the other multiplet partners may be generated. The multiplet is then constructed by acting on the prime state with the  $G = 1$  operators  $a_\varphi \equiv [Q_B, \varphi]$  and  $a_{X^\mu} \equiv [Q_B, X^\mu]$ . These operators are manifestly BRST invariant, but they are not

BRST trivial since  $\varphi$  and  $X^\mu$  are not primary conformal fields [12,10]. The  $a_\varphi$  and  $a_{X^\mu}$  operators, which have conformal weight 0, act by normal ordering with the operator that creates the physical state, giving rise in general to new physical states with ghost number boosted by 1. By repeatedly acting with  $a_\varphi$  and  $a_{X^\mu}$ , the entire multiplet associated with a given prime state is generated. In the present paper, the only such physical states that we shall be concerned with are those obtained by acting just once on a prime state, *i.e.* states with ghost number  $G = 0$ . For convenience, we summarise the prime states that we shall be using in this paper in a table:

	$G$	$L_0^{\text{eff}}$	$\beta$
$\ell = 0$	3	15/16	$-Q$
	3	1	$-6Q/7, -8Q/7$
$\ell = 1$	2	15/16	$-3Q/7$
	2	1/2	$-4Q/7$
$\ell = 2$	2	1/2	$-2Q/7$
$\ell = 3$	2	1	0

Table 1.

Every BRST non-trivial physical state has non-zero norm, in the sense that it has a non-vanishing inner product with some other physical state, *i.e.* its conjugate. We shall defer a more detailed discussion of this until the next section. For now, we just remark that the conjugate of a state with ghost number  $G$  and momentum  $(\beta, p_\mu)$  occurs at ghost number  $-G + 2$  and, because of the background charges, at momentum  $(-\beta - 2Q, -p_\mu - 2a_\mu)$ .

To conclude this section, we remark that *all* the BRST non-trivial physical states are highest-weight states with conformal dimension 0 with respect to the total energy-momentum tensor  $T^{\text{tot}} \equiv T + T_{\text{gh}}$ . This follows because, as in ordinary string theory [12], any physical state  $|\chi\rangle$  is certainly an eigenstate of  $L_0^{\text{tot}}$ , and so we have  $L_0^{\text{tot}}|\chi\rangle = \lambda|\chi\rangle = \{Q_B, b_0\}|\chi\rangle = Q_B b_0|\chi\rangle$ . Thus if  $|\chi\rangle$  is BRST non-trivial, it must be that  $\lambda = 0$ , since otherwise we would have  $|\chi\rangle = \lambda^{-1} Q_B b_0|\chi\rangle$ . \*

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\* It is not true, however, that there exists in general a basis for the physical states such that they are eigenstates of  $W_0^{\text{tot}}$ , even though  $Q_B$ ,  $L_0^{\text{tot}}$  and  $W_0^{\text{tot}}$  commute. To see this, suppose that  $|\chi_i\rangle$  denotes *all* the states at given  $\ell$  and  $G$  that are annihilated by  $Q_B$ . Since  $Q_B W_0^{\text{tot}}|\chi_i\rangle = 0$ , it follows that we can write  $W_0^{\text{tot}}|\chi_i\rangle = \sum_j a_{ij}|\chi_j\rangle$ . The problem is that  $a_{ij}$  is non-hermitean, and cannot always be diagonalised, even though  $W_0^{\text{tot}}$  is an hermitean operator. (This is because  $a_{ij}$  are not the matrix elements of  $W_0^{\text{tot}}$  with respect to the proper  $SL(2, C)$ -invariant inner product.) We have found explicit examples of physical states at  $\ell = 4$  and  $G = 0$  where the diagonalisation of  $a_{ij}$  is impossible. It is true, however, that  $W_0^{\text{tot}}$  on any physical state gives a BRST-trivial state, since  $W_0^{\text{tot}}|\chi\rangle = \{Q_B, \beta_0\}|\chi\rangle = Q_B \beta_0|\chi\rangle$ . It follows from this that the matrix of  $W_0^{\text{tot}}$  inner products with respect to the proper  $SL(2, C)$ -invariant inner product is not merely hermitean, but actually zero.

### 3. Interactions in the $W_3$ string

#### 3.1 Introduction

As in ordinary string theory at the first-quantised level [17], interactions for the  $W_3$  string have to be introduced by hand. The guiding principle for the construction of the interaction terms is that they should be gauge invariant. In other words, we need to build BRST invariant scattering amplitudes. These can be constructed from correlation functions of the BRST invariant physical operators that we described in section 2. However, as we shall explain below, there are no interactions among physical states with only standard ghost structure. The main point of this paper is to show that interactions can in fact occur in the  $W_3$  string, but that they necessarily involve states of non-standard ghost structure.

For a correlation function of operators to be non-vanishing, two necessary conditions must be satisfied. The first is that the operators must have the correct total ghost number; the second is that they must satisfy momentum conservation.

In addition to the standard ghost vacuum  $|--\rangle$  defined in (2.7), there are three more ghost vacua  $|+-\rangle = c_0|--\rangle$ ,  $| -+\rangle = \gamma_0|--\rangle$  and  $|++\rangle = c_0\gamma_0|--\rangle$  which are degenerate in energy with  $|--\rangle$ . Since we therefore have that  $|--\rangle = \beta_0 b_0|++\rangle$ , it follows that  $\langle --|--\rangle = \langle ++|++\rangle = 0$ , *etc.* The basic non-vanishing inner product is

$$\begin{aligned} 1 &= \langle ++|--\rangle = \langle 0|c_{-1}c_0c_1\gamma_{-2}\gamma_{-1}\gamma_0\gamma_1\gamma_2|0\rangle \\ &= \frac{1}{576}\langle 0|\partial^2 c\partial c c\partial^4 \gamma\partial^3 \gamma\partial^2 \gamma\partial \gamma \gamma|0\rangle. \end{aligned} \quad (3.1)$$

Note that in particular the total ghost number of the operators in a non-vanishing correlator must be  $3 + 5 = 8$ .

We find it convenient for calculating correlation functions to bosonise the ghosts as follows:

$$\begin{aligned} b &= e^{-i\sigma}; & c &= e^{i\sigma}, \\ \beta &= e^{-i\rho}; & \gamma &= e^{i\rho}, \end{aligned} \quad (3.2)$$

where  $\sigma$  and  $\rho$  are real scalars with the operator-product expansions  $\sigma(z)\sigma(w) \sim -\log(z-w)$  and  $\rho(z)\rho(w) \sim -\log(z-w)$ . It is straightforward to see that

$$\partial^n c \partial^{n-1} c \cdots \partial c c = n!(n-1)!\cdots 1 e^{i(n+1)\sigma}, \quad (3.3)$$

and similarly for  $\gamma$ . Thus (3.1) becomes

$$\langle 0|e^{3i\sigma} e^{5i\rho}|0\rangle = 1. \quad (3.4)$$

In addition to having the ghost structure given above, non-vanishing correlators must also satisfy momentum conservation. Owing to the presence of the background charges, we



must have  $\sum_{i=1}^N p_i^\mu = -2a^\mu$  in the effective spacetime together with

$$\sum_{i=1}^N \beta_i = -2Q \quad (3.5)$$

in the  $\varphi$  direction, in order to have a non-vanishing  $N$ -point function. For states of continuous spacetime momentum  $p_\mu$ , as indeed we have in the multi-scalar  $W_3$  string, momentum conservation in the  $X^\mu$  directions can be straightforwardly satisfied. However, as we saw in section 2, the momentum  $\beta$  in the  $\varphi$  direction can only take specific frozen values in physical states. Thus it is in general non-trivial to satisfy momentum conservation in the  $\varphi$  direction.

Note *en passant* that since all the physical operators are primary fields with dimension zero with respect to  $T + T_{\text{gh}}$ , the conformal prefactors in all correlation functions will be trivial.

### 3.2 The two-point function

We begin our detailed discussion of correlation functions by considering the two-point function. It is this that defines the inner product, and hence the meaning of conjugation of any state. Physical states with standard ghost structure provide a good example for this discussion. One can see from (2.6), (2.7) and (2.10) that the  $G = 3$  physical operator describing a standard state is of the form

$$\begin{aligned} \mathbf{V}_\Delta^3[\beta, p] &= c \partial \gamma \gamma e^{\beta \varphi} P(\partial X) e^{ip \cdot X} \\ &= e^{i\sigma} e^{2i\rho} e^{\beta \varphi} P(\partial X) e^{ip \cdot X} , \end{aligned} \quad (3.6)$$

where  $\beta$  satisfies (2.11),  $P(\partial X) e^{ip \cdot X} |0\rangle = |\text{phys}\rangle_{\text{eff}}$  satisfies (2.12), and  $\Delta$  is given below (2.12). The  $G = 5$  operator conjugate to (3.6) that acts on  $\langle 0|$  to create the conjugate state has the form

$$\begin{aligned} \mathbf{V}_{\Delta'}^5[\beta', p'] &= \partial c c \partial^2 \gamma \partial \gamma \gamma e^{\beta' \varphi} P(\partial X) e^{ip' \cdot X} \\ &= 2 e^{2i\sigma} e^{3i\rho} e^{\beta' \varphi} P(\partial X) e^{ip' \cdot X} . \end{aligned} \quad (3.7)$$

The physical-state conditions imply that  $\beta'$  must satisfy the same cubic equation (2.11) as does  $\beta$ , and  $P(\partial X) e^{ip' \cdot X}$  gives rise to a state satisfying (2.12). From (3.5) we see that the solution  $\beta = -Q$  of (2.11) is self-conjugate, and that the solutions  $\beta = -\frac{6}{7}Q$  and  $\beta = -\frac{8}{7}Q$  are conjugate to each other [4,5,8]. The inner product between the above states, when we choose  $\beta'$  so as to satisfy (3.5), is

$$\begin{aligned} &\langle 0 | \mathbf{V}_{\Delta'}^5[-2Q - \beta, -2a - p](z_1) \mathbf{V}_\Delta^3[\beta, p](z_2) | 0 \rangle \\ &= 2 \langle 0 | e^{2i\sigma} e^{3i\rho} e^{(-2Q - \beta)\varphi}(z_1) e^{i\sigma} e^{2i\rho} e^{\beta\varphi}(z_2) | 0 \rangle \langle \text{phys}' | \text{phys} \rangle_{\text{eff}} \\ &= -2 z_{12}^2 z_{12}^6 z_{12}^{\beta(\beta+2Q)} z_{12}^{-2\Delta} . \end{aligned} \quad (3.8)$$

In the last line we have used Wick's theorem and written the contributions from  $\sigma$ ,  $\rho$ ,  $\varphi$  and  $|\text{phys}\rangle_{\text{eff}}$  in that order. We use the standard notation  $z_{ij} \equiv z_i - z_j$ . For each solution given by (2.11) and its corresponding value of  $\Delta$  given below (2.12), we see that the exponent of  $z_{12}$  is zero, and thus the inner product is just a constant, as it must be for the two-point function of conformal-weight 0 fields. (The precise value of the constant is determined by the normalisations of the physical operators. Of course because the inner product is off-diagonal one must, as in ordinary string theory, truncate the states to obtain a positive-definite metric on the Hilbert space [8]. This is usually done by identifying states and their conjugates.) The calculation of inner products of states involving non-standard ghost structures proceeds in a similar fashion.

### 3.3 The three-point function

In ordinary string theory the three-point function defines the basic interaction vertex of the theory [17]. This is also the case for the  $W_3$  string, as we shall explain. There are, however, differences and subtleties that do not arise for the ordinary string.

Three-point functions are identically zero if all the external states have the standard ghost structure (2.6). This follows from the fact that neither of the two necessary conditions discussed in subsection 3.1 is satisfied in this case. To see this, we note that the ghost number of the physical operator that creates a standard state is  $G = 3$ , and thus the product of three such operators has ghost number  $G = 9$ . From (3.1) it then follows that the inner product in the ghost sector gives zero. Note that since the conjugate operators have ghost number  $G = 5$ , they cannot remedy this problem. There is also another reason why these three-point functions vanish. Since the allowed values of  $\beta$  momentum are in this case  $\beta = -Q$ ,  $\beta = -\frac{6}{7}Q$  or  $\beta = -\frac{8}{7}Q$ , it follows that three of them cannot be combined so as to satisfy the momentum-conservation law (3.5). Similar arguments show that all higher  $N$ -point functions of physical states with the standard ghost structure, or their conjugates, are identically zero.

This apparent difficulty of introducing interactions in  $W_3$  string theory can be resolved by considering three-point functions of physical states with non-standard ghost structure. These states circumvent both of the difficulties described above, since, as we saw in section 2, they occur with lower ghost number and less negative  $\beta$  momentum, as compared with the states of standard ghost structure. With the physical states that we have described in section 2 and the appendix, there are many non-vanishing three-point functions. We shall begin by presenting a detailed computation of one example and shall then give the results for many others.

The easiest way to identify a possible non-vanishing three-point function is first to ensure that the three states have momenta satisfying the momentum-conservation law (3.5). (As explained earlier, the only non-trivial requirement comes from the conservation equation for the  $\beta$  momentum.) Since all states in a multiplet have the same momentum, it suffices to look just at the prime state in each multiplet. Having taken care of momentum conservation,

we must also ensure that the ghost structure of the product of operators leads to the non-vanishing ghost correlation function (3.1). In particular, this means that the total ghost number of the operators must be 8. If the sum of ghost numbers for the three prime states is greater than 8, then the three-point function will be zero. If the sum of ghost numbers equals 8, then this is a good candidate for a non-vanishing three-point function. If the sum is less than 8, then the ghost numbers can be boosted by acting with  $a_\varphi$  or  $a_{X^\mu}$  to generate a higher ghost number member of a multiplet. This again can lead to a non-vanishing three-point function.

Let us consider the following example in detail. For the three prime states we choose one to be a tachyon state with standard ghost structure, with  $\beta = -Q$ ; this corresponds to the operator  $\mathbf{V}_\Delta^G[\beta, p] = \mathbf{V}_{15/16}^3[-Q, p_1]$  as given in (A.3). The two remaining states that we choose have non-standard ghost structure, and occur at level  $\ell = 1$ . One has  $\beta = -\frac{3}{7}Q$ , and corresponds to the operator  $\mathbf{V}_{15/16}^2[-\frac{3}{7}Q, p_2]$  given in (A.7), and the other has  $\beta = -\frac{4}{7}Q$  and corresponds to the operator  $\mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p_3]$  given in (A.8). We emphasise that even though these are level 1 states, they are tachyonic from the point of view of the matter, since the excitations are purely ghostly. Clearly the  $\beta$  momenta satisfy the conservation condition (3.5). However, the ghost numbers add up to 7, and so we must boost this to 8 by acting on one of the prime states with a linear combination of  $a_\varphi$  and  $a_{X^\mu}$ . We choose to boost the ghost number of the second state; accordingly we take the operator  $\mathbf{W}_\Delta^G[\beta, p] = \mathbf{W}_{15/16}^3[-\frac{3}{7}Q, p_2]$  given in (A.9).

Not all the terms in these three states can combine to give the correct ghost structure (3.4); only two of the three terms in  $\mathbf{W}_{15/16}^3[-\frac{3}{7}Q, p_2]$ , and one of the two terms in  $\mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p_3]$  contribute. We find it convenient to compute the ghost and  $\varphi$  part separately from the effective matter part, since all the states that we consider in this paper factorise in this way. Thus we have for the ghost and  $\varphi$  part

$$\begin{aligned} & \langle 0 | (e^{i\sigma} e^{2i\rho} e^{-Q\varphi})(z_1) (11e^{i\sigma} \partial e^{2i\rho} e^{-\frac{3}{7}Q\varphi} + 16e^{i\sigma} e^{2i\rho} \partial e^{-\frac{3}{7}Q\varphi})(z_2) (e^{i\sigma} e^{i\rho} e^{-\frac{4}{7}Q\varphi})(z_3) | 0 \rangle \\ &= -z_{12} z_{13} z_{23} [11\partial_2(z_{12}^4 z_{13}^2 z_{23}^2) z_{12}^{-21/8} z_{13}^{-7/2} z_{23}^{-3/2} + 16z_{12}^4 z_{13}^2 z_{23}^2 \partial_2(z_{12}^{-21/8} z_{13}^{-7/2} z_{23}^{-3/2})] \\ &= -2 z_{12}^{11/8} (z_{13} z_{23})^{1/2} , \end{aligned} \tag{3.9}$$

where  $\partial_i \equiv \partial/\partial z_i$ . The effective matter part is easier to compute, since the operators are all tachyonic. For this, we have

$$\begin{aligned} \langle e^{ip_1 \cdot X(z_1)} e^{ip_2 \cdot X(z_2)} e^{ip_3 \cdot X(z_3)} \rangle &= z_{12}^{p_1 \cdot p_2} z_{13}^{p_1 \cdot p_3} z_{23}^{p_2 \cdot p_3} \\ &= z_{12}^{-11/8} (z_{13} z_{23})^{-1/2} , \end{aligned} \tag{3.10}$$

where in deriving the last line we have used the relation

$$p_1 \cdot p_2 = \Delta_3 - \Delta_1 - \Delta_2 = \frac{1}{2} - \frac{15}{16} - \frac{15}{16} = -\frac{11}{8} , \quad \text{etc} , \tag{3.11}$$

which follows from the momentum-conservation law (3.5). Putting the factors (3.9) and (3.10) together, we finally obtain the three-point function

- $L_0^{\text{eff}} = \{\frac{15}{16}, \frac{15}{16}, \frac{1}{2}\}$ :

$$\langle 0 | \mathbf{V}_{15/16}^3[-Q, p_1](z_1) \mathbf{W}_{15/16}^3[-\frac{3}{7}Q, p_2](z_2) \mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p_3](z_3) | 0 \rangle = -2 . \quad (3.12)$$

That this three-point function is a constant is a consequence of the fact that it is the correlator of three primary fields of dimension zero with respect to the total energy-momentum tensor  $T + T_{\text{gh}}$ . The important point is that the constant is non-zero, showing that there are indeed interactions in the  $W_3$  string.

There are two three-point functions that are closely related to the one that we have just calculated. These correspond to the cases where we boost the ghost number of the first or the third operator instead of the second. The computations are similar and we just give the results:

$$\begin{aligned} \langle 0 | \mathbf{W}_{15/16}^4[-Q, p_1](z_1) \mathbf{V}_{15/16}^2[-\frac{3}{7}Q, p_2](z_2) \mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p_3](z_3) | 0 \rangle &= -2 , \\ \langle 0 | \mathbf{V}_{15/16}^3[-Q, p_1](z_1) \mathbf{V}_{15/16}^2[-\frac{3}{7}Q, p_2](z_2) \mathbf{W}_{1/2}^3[-\frac{4}{7}Q, p_3](z_3) | 0 \rangle &= -2 . \end{aligned} \quad (3.13)$$

The fact that these two three-point functions also turn out to be non-zero is a first sign of a pattern that we shall encounter in all the correlation functions we compute, namely, that when it is necessary to boost the total ghost number in a correlation function in order to reach 8, it does not seem to matter which operator is boosted. (Recall that boosting an operator means acting with a linear combination of  $a_\varphi$  and  $a_{X^\mu}$ , giving another member in the same multiplet, but with ghost number increased by one.)

Another interesting three-point function to compute is one with three  $L_0^{\text{eff}} = 1$  operators. The first two operators have standard ghost structure with  $\beta$ -momenta  $-\frac{6}{7}Q$  and  $-\frac{8}{7}Q$ . The third is a level  $\ell = 3$  operator with non-standard ghost structure and  $\beta = 0$ . Their explicit expressions are given by (A.2), (A.1) and (A.12). The total ghost number of these three operators is already 8, so no boosting is needed. The result turns out to be

- $L_0^{\text{eff}} = \{1, 1, 1\}$ :

$$\langle 0 | \mathbf{V}_1^3[-\frac{6}{7}Q, p_1](z_1) \mathbf{V}_1^3[-\frac{8}{7}Q, p_2](z_2) \mathbf{V}_1^2[0, p_3](z_3) | 0 \rangle = -2 . \quad (3.14)$$

In the above calculation we have used

$$\beta \partial \gamma \gamma(z) = : e^{-i\rho} e^{2i\rho} : (z) = \frac{1}{2\pi i} \oint \frac{dw}{w-z} e^{-i\rho(w)} e^{2i\rho(z)} , \quad (3.15)$$

and similarly

$$(\partial \varphi)^2(z) = \frac{1}{2\pi i} \oint \frac{dw}{w-z} \left( e^{-i\varphi(w)} \partial e^{i\varphi(z)} - \partial e^{-i\varphi(w)} e^{i\varphi(z)} \right) . \quad (3.16)$$

The relevant terms where these factors appear can then be evaluated as four-point functions with a contour integral.

From the physical operators given in the appendix, there are several more three-point functions that satisfy momentum conservation and have the correct total ghost number. Classifying them by the  $L_0^{\text{eff}}$  values of the operators,<sup>\*</sup> the results are as follows:

- $L_0^{\text{eff}} = \{\frac{15}{16}, \frac{15}{16}, 1\}$ :

$$\langle 0 | \mathbf{W}_{15/16}^3[-\frac{3}{7}Q, p_1](z_1) \mathbf{V}_{15/16}^2[-\frac{3}{7}Q, p_2](z_2) \mathbf{V}_1^3[-\frac{8}{7}Q, p_3](z_3) | 0 \rangle = 4 . \quad (3.17)$$

- $L_0^{\text{eff}} = \{\frac{15}{16}, \frac{15}{16}, 1\}$ :

$$\langle 0 | \mathbf{V}_{15/16}^3[-Q, p_1](z_1) \mathbf{V}_{15/16}^3[-Q, p_2](z_2) \mathbf{V}_1^2[0, p_3](z_3) | 0 \rangle = 1 . \quad (3.18)$$

- $L_0^{\text{eff}} = \{\frac{1}{2}, \frac{1}{2}, 1\}$ :

$$\langle 0 | \mathbf{W}_{1/2}^3[-\frac{4}{7}Q, p_1](z_1) \mathbf{V}_{1/2}^2[-\frac{2}{7}Q, p_2](z_2) \mathbf{V}_1^3[-\frac{8}{7}Q, p_3](z_3) | 0 \rangle = 2 . \quad (3.19)$$

- $L_0^{\text{eff}} = \{\frac{1}{2}, \frac{1}{2}, 1\}$ :

$$\langle 0 | \mathbf{W}_{1/2}^3[-\frac{4}{7}Q, p_1](z_1) \mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p_2](z_2) \mathbf{V}_1^3[-\frac{6}{7}Q, p_3](z_3) | 0 \rangle = 4 . \quad (3.20)$$

- $L_0^{\text{eff}} = \{1, 1, \frac{1}{2}\}$ :

$$\langle 0 | \mathbf{V}_1^3[-\frac{6}{7}Q, p_1](z_1) \mathbf{V}_1^3[-\frac{6}{7}Q, p_2](z_2) \mathbf{V}_{1/2}^2[-\frac{2}{7}Q, p_3](z_3) | 0 \rangle = 0 . \quad (3.21)$$

When the boosting of an operator was necessary in the above three-point functions, we have only shown the results for one specific choice of which operator to boost. Similar expressions are obtained when a different operator is boosted. No non-vanishing three-point functions where all three physical operators have non-standard ghost structure can be constructed with the examples given in the appendix.

The vanishing of the three-point function (3.21) emerges only after a computation. The result is indicative of a relation with the fusion rules of the Ising model, as indeed are the

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<sup>\*</sup> The reason for classifying them in this way is that, as we shall argue in section 4, physical operators with the same  $L_0^{\text{eff}}$  value can be viewed as equivalent, even though they may have different ghost structures and  $\varphi$  dependence.

results of all the other three-point functions given above. We shall discuss this further in section 4.

### 3.4 The four-point function

By studying the poles of the four-point functions of the  $W_3$  string, one learns about the mass spectrum of theory. There are several four-point functions that we can calculate using the physical operators given in the appendix. We shall begin by describing the procedure for computing four-point functions in the  $W_3$  string, and illustrate it in detail with an example. Then, we shall present the results for all the other four-point functions.

By making use of the  $SL(2, C)$  invariance of the vacuum  $|0\rangle$ , three of the worldsheet coordinates of the physical operators in a four-point function may be fixed. As usual, we choose to set  $z_1 = \infty$ ,  $z_2 = 1$  and  $z_4 = 0$ . As in ordinary string theory the coordinate  $z_3$  should then be integrated from 0 to 1, giving a scattering amplitude that is independent of the positions of insertion of the physical operators. The physical operators have conformal dimension 0 with respect to the total energy-momentum tensor  $T + T_{\text{gh}}$ . Therefore in order to preserve the conformal covariance of the theory, the dimension of the physical operator  $V(z_3)$  inserted at  $z_3$  must be increased by 1 so that we can integrate over an operator of dimension 1, as we must for an invariant result. This may be achieved by making the replacement

$$V(z_3) \rightarrow \frac{1}{2\pi i} \oint_{z_3} dw b(w) V(z_3) , \quad (3.22)$$

where the subscript on the contour integral indicates that it is to be evaluated around a closed path enclosing  $z_3$ . The above procedure not only preserves the projective structure (*i.e.*  $SL(2, C)$  covariance) but also gives a result that is invariant under the BRST transformations generated by (2.1). This whole construction is parallel to the one used in ordinary string theory. It also admits an immediate generalisation to higher-point functions.

As in the case of three-point functions, a four-point function will be zero unless the ghost numbers of the operators (after the  $b$  contour integral) add up to 8 and momentum conservation is satisfied. If these necessary conditions are satisfied then it becomes a matter of computation to determine the result. We shall now carry this out for the following example. Let us take four physical operators that all correspond to physical states with  $L_0^{\text{eff}} = \frac{1}{2}$ . We can satisfy the momentum conservation by choosing three of them to be level 1 states with non-standard ghost structure and  $\beta = -\frac{4}{7}Q$ , given by (A.8), and the other to be a level 2 state with non-standard ghost structure and  $\beta = -\frac{2}{7}Q$ , given by (A.11). Since they already have the correct total ghost number, to insert the  $b$  contour integral we need to boost one of the operators to restore the total ghost number to 8. Without loss of generality, we shall

boost the operator at  $z_4$ . (As we shall discuss later, it does not matter which operator we choose to boost.) Thus the four-point function for these four operators takes the form:

$$\int dz_3 \oint_{z_3} \frac{dw}{2\pi i} \times \langle 0 | \mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p_1](z_1) \mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p_2](z_2) b(w) \mathbf{V}_{1/2}^2[-\frac{2}{7}Q, p_3](z_3) \mathbf{W}_{1/2}^3[-\frac{4}{7}Q, p_4](z_4) | 0 \rangle , \quad (3.23)$$

where the boosted operator  $\mathbf{W}_{1/2}^3[-\frac{4}{7}Q, p_4](z_4)$  is given in (A.10).

This four-point function may be factorised as a product of the ghost plus  $\varphi$  part, and a matter part. For the ghost plus  $\varphi$  part, we obtain

$$\begin{aligned} & \langle 0 | (e^{i\sigma} e^{i\rho} e^{-\frac{4}{7}Q\varphi})(z_1) (e^{i\sigma} e^{i\rho} e^{-\frac{4}{7}Q\varphi})(z_2) ( -\frac{3}{2} \partial_3 e^{i\rho} e^{-\frac{2}{7}Q\varphi} - 2e^{i\rho} \partial_3 e^{-\frac{2}{7}Q\varphi} ) (z_3) \\ & (10e^{i\sigma} \partial_4 e^{2i\rho} e^{-\frac{4}{7}Q\varphi} + 12e^{i\sigma} e^{2i\rho} \partial_4 e^{-\frac{4}{7}Q\varphi} ) (z_4) | 0 \rangle \end{aligned} \quad (3.24)$$

$$\begin{aligned} & = -15\mathcal{A}_\sigma \partial_3 \partial_4 \mathcal{A}_\rho \mathcal{A}_\varphi - 18\mathcal{A}_\sigma \partial_3 \mathcal{A}_\rho \partial_4 \mathcal{A}_\varphi - 20\mathcal{A}_\sigma \partial_4 \mathcal{A}_\rho \partial_3 \mathcal{A}_\varphi - 24\mathcal{A}_\sigma \mathcal{A}_\rho \partial_3 \partial_4 \mathcal{A}_\varphi \\ & = 2(z_{12} z_{14} z_{24})^{2/3} (z_{13} z_{23} z_{34})^{-1/3} x^{-2/3} (1-x)^{-2/3} (1-x+x^2) . \end{aligned}$$

Here  $\mathcal{A}_\sigma$ ,  $\mathcal{A}_\rho$  and  $\mathcal{A}_\varphi$  denote the contractions from the  $\sigma$ ,  $\rho$  and  $\varphi$  fields, and are given by

$$\begin{aligned} \mathcal{A}_\sigma &= z_{12} z_{14} z_{24} , \\ \mathcal{A}_\rho &= z_{12} z_{13} z_{23} z_{14}^2 z_{24}^2 z_{34}^2 , \\ \mathcal{A}_\varphi &= z_{12}^{-2} z_{13}^{-1} z_{14}^{-2} z_{23}^{-1} z_{24}^{-2} z_{34}^{-1} . \end{aligned} \quad (3.25)$$

In the last line of (3.24) we have extracted the conformal prefactor, and written the remainder in terms of the invariant cross-ratio

$$x = \frac{z_{12} z_{34}}{z_{13} z_{24}} . \quad (3.26)$$

In order to compute the effective matter part of the four-point function, we introduce the Mandelstam variables  $s$ ,  $t$  and  $u$ :

$$\begin{aligned} s &\equiv -(p_1 + p_2)^2 - 2a \cdot (p_1 + p_2) = -2p_1 \cdot p_2 - 2\Delta_1 - 2\Delta_2 , \\ t &\equiv -(p_2 + p_3)^2 - 2a \cdot (p_2 + p_3) = -2p_2 \cdot p_3 - 2\Delta_2 - 2\Delta_3 , \\ u &\equiv -(p_1 + p_3)^2 - 2a \cdot (p_1 + p_3) = -2p_1 \cdot p_3 - 2\Delta_1 - 2\Delta_3 . \end{aligned} \quad (3.27)$$

Their sum is given by

$$s + t + u = -2(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4) . \quad (3.28)$$

In our present calculation, where  $\Delta_i = \frac{1}{2}$ , we have  $s + t + u = -4$ . The matter part of the four-point function (3.23) gives

$$\begin{aligned} \langle e^{ip_1 \cdot X(z_1)} e^{ip_2 \cdot X(z_2)} e^{ip_3 \cdot X(z_3)} e^{ip_4 \cdot X(z_4)} \rangle &= z_{12}^{p_1 \cdot p_2} z_{13}^{p_1 \cdot p_3} z_{14}^{p_1 \cdot p_4} z_{23}^{p_2 \cdot p_3} z_{24}^{p_2 \cdot p_4} z_{34}^{p_3 \cdot p_4} \\ &= (z_{12} z_{13} z_{14} z_{23} z_{24} z_{34})^{-1/3} x^{-s/2-2/3} (1-x)^{-t/2-2/3} , \end{aligned} \quad (3.29)$$

where again we have extracted the appropriate conformal prefactor. Combining (3.24) and (3.29), setting  $z_1 = \infty$ ,  $z_2 = 1$  and  $z_4 = 0$ , and integrating  $x = z_3$  from 0 to 1, we finally obtain for the scattering amplitude (3.23)

- $\underline{L_0^{\text{eff}} = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}}:$

$$\int \oint_{z_3} \langle 0 | \mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p_1](z_1) \mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p_2](z_2) b(w) \mathbf{V}_{1/2}^2[-\frac{2}{7}Q, p_3](z_3) \mathbf{W}_{1/2}^3[-\frac{4}{7}Q, p_4](z_4) | 0 \rangle$$

$$= 2 \int_0^1 dx x^{-s/2-2} (1-x)^{-t/2-2} (1-x+x^2) \quad (3.30)$$

$$= 2 \frac{\Gamma(-s/2-1)\Gamma(-t/2)}{\Gamma(-s/2-t/2-1)} + 2 \frac{\Gamma(-s/2+1)\Gamma(-t/2-1)}{\Gamma(-s/2-t/2)} . \quad (3.31)$$

Here, and in subsequent expressions, the integrals at the front of the first line denote precisely the integrations given in (3.23). We shall defer detailed discussion of this result, and of the other four-point functions we shall compute, until section 4.

There are several other four-point functions that satisfy momentum conservation and have the correct ghost number, and we now list the results. Just as for the three-point functions, we classify them by the  $L_0^{\text{eff}}$  values of the physical operators.

- $\underline{L_0^{\text{eff}} = \{\frac{15}{16}, \frac{15}{16}, \frac{1}{2}, \frac{1}{2}\}}:$

$$\int \oint_{z_3} \langle 0 | \mathbf{V}_{15/16}^2[-\frac{3}{7}Q, p_1](z_1) \mathbf{V}_{15/16}^2[-\frac{3}{7}Q, p_2](z_2) b(w) \mathbf{W}_{1/2}^3[-\frac{4}{7}Q, p_3](z_3) \mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p_4](z_4) | 0 \rangle$$

$$= -2 \int_0^1 dx x^{-s/2-2} (1-x)^{-t/2-31/16} (x-2) \quad (3.32)$$

$$= -2 \frac{\Gamma(-s/2)\Gamma(-t/2-15/16)}{\Gamma(-s/2-t/2-15/16)} + 4 \frac{\Gamma(-s/2-1)\Gamma(-t/2-15/16)}{\Gamma(-s/2-t/2-31/16)} . \quad (3.33)$$

- $\underline{L_0^{\text{eff}} = \{\frac{15}{16}, \frac{15}{16}, \frac{1}{2}, \frac{1}{2}\}}:$

$$\int \oint_{z_3} \langle 0 | \mathbf{V}_{15/16}^3[-Q, p_1](z_1) \mathbf{V}_{15/16}^2[-\frac{3}{7}Q, p_2](z_2) b(w) \mathbf{V}_{1/2}^2[-\frac{2}{7}Q, p_3](z_3) \mathbf{V}_{1/2}^2[-\frac{2}{7}Q, p_4](z_4) | 0 \rangle$$

$$= 0 . \quad (3.34)$$

- $\underline{L_0^{\text{eff}} = \{1, \frac{15}{16}, \frac{15}{16}, 1\}}:$

$$\int \oint_{z_3} \langle 0 | \mathbf{V}_1^3[-\frac{8}{7}Q, p_1](z_1) \mathbf{V}_{15/16}^2[-\frac{3}{7}Q, p_2](z_2) b(w) \mathbf{V}_{15/16}^2[-\frac{3}{7}Q, p_3](z_3) \mathbf{V}_1^2[0, p_4](z_4) | 0 \rangle$$

$$= -2 \int_0^1 dx x^{-s/2-31/16} (1-x)^{-t/2-2} \quad (3.35)$$

$$= -2 \frac{\Gamma(-s/2-15/16)\Gamma(-t/2-1)}{\Gamma(-s/2-t/2-31/16)} . \quad (3.36)$$



- $L_0^{\text{eff}} = \{\frac{15}{16}, \frac{15}{16}, 1, \frac{1}{2}\}$ :

$$\begin{aligned} \int \oint_{z_3} \langle 0 | \mathbf{V}_{15/16}^3[-Q, p_1](z_1) \mathbf{V}_{15/16}^2[-\frac{3}{7}Q, p_2](z_2) b(w) \mathbf{V}_1^2[0, p_3](z_3) \mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p_4](z_4) | 0 \rangle \\ = \int_0^1 dx x^{-s/2-3/2} (1-x)^{-t/2-31/16} \end{aligned} \quad (3.37)$$

$$= \frac{\Gamma(-s/2-1/2)\Gamma(-t/2-15/16)}{\Gamma(-s/2-t/2-23/16)} . \quad (3.38)$$

- $L_0^{\text{eff}} = \{1, \frac{1}{2}, 1, \frac{1}{2}\}$ :

$$\begin{aligned} \int \oint_{z_3} \langle 0 | \mathbf{V}_1^3[-\frac{6}{7}Q, p_1](z_1) \mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p_2](z_2) b(w) \mathbf{V}_1^2[0, p_3](z_3) \mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p_4](z_4) | 0 \rangle \\ = -2 \int_0^1 dx x^{-s/2-3/2} (1-x)^{-t/2-3/2} \end{aligned} \quad (3.39)$$

$$= -2 \frac{\Gamma(-s/2-1/2)\Gamma(-t/2-1/2)}{\Gamma(-s/2-t/2-1)} . \quad (3.40)$$

- $L_0^{\text{eff}} = \{1, \frac{1}{2}, \frac{1}{2}, 1\}$ :

$$\begin{aligned} \int \oint_{z_3} \langle 0 | \mathbf{V}_1^3[-\frac{8}{7}Q, p_1](z_1) \mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p_2](z_2) b(w) \mathbf{V}_{1/2}^2[-\frac{2}{7}Q, p_3](z_3) \mathbf{V}_1^2[0, p_4](z_4) | 0 \rangle \\ = \int_0^1 dx x^{-s/2-3/2} (1-x)^{-t/2-2} \end{aligned} \quad (3.41)$$

$$= \frac{\Gamma(-s/2-1/2)\Gamma(-t/2-1)}{\Gamma(-s/2-t/2-3/2)} . \quad (3.42)$$

- $L_0^{\text{eff}} = \{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$ :

$$\begin{aligned} \int \oint_{z_3} \langle 0 | \mathbf{V}_1^3[-\frac{6}{7}Q, p_1](z_1) \mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p_2](z_2) b(w) \mathbf{V}_{1/2}^2[-\frac{2}{7}Q, p_3](z_3) \mathbf{V}_{1/2}^2[-\frac{2}{7}Q, p_4](z_4) | 0 \rangle \\ = 0 . \end{aligned} \quad (3.43)$$

- $L_0^{\text{eff}} = \{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$ :

$$\begin{aligned} \int \oint_{z_3} \langle 0 | \mathbf{V}_1^3[-\frac{8}{7}Q, p_1](z_1) \mathbf{V}_{1/2}^2[-\frac{2}{7}Q, p_2](z_2) b(w) \mathbf{V}_{1/2}^2[-\frac{2}{7}Q, p_3](z_3) \mathbf{V}_{1/2}^2[-\frac{2}{7}Q, p_4](z_4) | 0 \rangle \\ = 0 . \end{aligned} \quad (3.44)$$

In the four-point functions (3.30) and (3.32), where it is necessary to boost the total ghost number, we have in each case made a specific choice about which operator to boost. We have checked in several examples that the result does not depend on which particular operator is chosen. We expect this to be a general feature of all correlation functions.

The four-point functions (3.34), (3.43) and (3.44) turn out to be zero as a result of non-trivial computations. The vanishing of (3.34) can be understood from the underlying three-point functions, *i.e.* that the intermediate state in the four-point function does not carry a  $\beta$  momentum for which there could be a non-vanishing three-point function with the two external states for that particular channel. This suggests that the three-point function is indeed the basic interaction vertex of the  $W_3$  string. The vanishing of the four-point functions (3.43) and (3.44) cannot be explained in this way. However, as we shall discuss in section 4, it is indicative of a relation between the  $W_3$  string and the Ising model.

## 4. Crossing, duality and the Ising model

Having obtained several non-vanishing three-point and four-point functions in the previous section, we now turn to a discussion of the significance of these results. We shall first discuss the crossing and duality properties of these correlation functions. Then we shall investigate their relation with the fusion rules of the Ising model.

### 4.1 Crossing properties

One of the fundamental properties of the ordinary bosonic string is that the four-point function is crossing symmetric [17]. For example, the four-point function for tachyons in the ordinary string takes the form

$$\int_0^1 dx x^{-s/2-2} (1-x)^{-t/2-2} , \quad (4.1)$$

and is invariant under the interchange of  $s$  and  $t$ . It should be stressed that this is a direct consequence of the  $SL(2, C)$  invariance of the vacuum  $|0\rangle$ . This invariance in general dictates how correlation functions with different orderings of the operators are related. In particular, for a case such as (4.1), where the operators are identical, the four-point function is *invariant* under  $s \leftrightarrow t$ , since the interchange of  $s$  and  $t$  corresponds precisely to the interchange of the second and fourth operators or of the first and third operators.

In the  $W_3$  string there are analogous consequences of the  $SL(2, C)$  invariance of the vacuum, again leading to relations among correlation functions with different orderings of the operators. From these, we can see that, as we mentioned in section 3, there are indications that the physical operators we are discussing in this paper are characterised by their  $L_0^{\text{eff}}$  values, regardless of their ghost structures and  $\varphi$  dependence. For example, the four-point

function (3.30) for four tachyons with  $L_0^{\text{eff}} = \frac{1}{2}$  is invariant under the interchange of  $s$  and  $t$ . So is the four-point function (3.39). Thus these four-point functions exhibit the crossing symmetries that would be expected if the second and fourth operators were identical and also if the first and the third operators were identical. The fact that these pairs of operators in (3.30) and (3.39) have identical  $L_0^{\text{eff}}$  values indeed provides evidence for the above suggestion that all operators with an effective spacetime interpretation and the same  $L_0^{\text{eff}}$  values should be regarded as “equivalent.”

Since the above-mentioned pairs of operators in the remaining four-point functions of section 3 have different  $L_0^{\text{eff}}$  values, they are not *invariant* under the “crossing transformations.” However, they do transform *covariantly*. For example, under the crossing transformation  $t \leftrightarrow u$ , implemented by the interchange of the third and fourth operators (and thus  $x \rightarrow -x/(1-x)$ ), the integrand in the four-point function (3.39) (including the measure  $dx$ ) transforms as

$$dx x^{-s/2-3/2}(1-x)^{-t/2-3/2} \rightarrow dx x^{-s/2-3/2}(1-x)^{-t/2-2} . \quad (4.2)$$

In fact, this result provides another illustration of the equivalence of different physical operators with the same  $L_0^{\text{eff}}$  value, since (4.2) leads precisely to (3.41).

## 4.2 Duality

Let us now turn to a discussion of duality. In the usual bosonic string, duality is the statement that the four-point amplitude can be expanded as an infinite sum over either  $s$ -channel or  $t$ -channel poles [17]. The form of the poles in the sum is identical in each case, since it is always the same set of intermediate string states that are exchanged in either channel.

The notion of duality exists for the  $W_3$  string, but it is slightly more subtle. In this case, four-point amplitudes can again be expanded as infinite sums over either  $s$ -channel or  $t$ -channel poles, as immediately follows from their expressions in terms of  $\Gamma$  functions. However, as we have seen, the  $W_3$  string contains different sectors corresponding to different  $L_0^{\text{eff}}$  values, and the set of intermediate states in one channel does not necessarily belong to the same sector as the set of intermediate states in another channel. Thus the form of the poles may be different in different channels. In particular, the masses of the exchanged particles in different channels may not be the same.

The four-point function (3.32) provides a nice illustration of this phenomenon. As can be seen from (3.33), if we expand in the  $s$ -channel, the poles correspond to the exchange of states from the  $L_0^{\text{eff}} = 1$  sector, with  $(\text{mass})^2 = 2n - 2$ , whilst if we expand in the  $t$ -channel, the poles correspond to the exchange of states from the  $L_0^{\text{eff}} = \frac{15}{16}$  sector, with  $(\text{mass})^2 = 2n - \frac{15}{8}$ . Similarly, in the four-point function (3.37), the exchanged states in the  $s$ -channel have  $L_0^{\text{eff}} = \frac{1}{2}$ , with  $(\text{mass})^2 = 2n - 1$ , whereas in the  $t$ -channel the exchanged states

have  $L_0^{\text{eff}} = \frac{15}{16}$ . Our expressions for four-point functions in subsection 3.4 make manifest the structure of poles in the  $s$ -channel and  $t$ -channel. Of course one can also look at the  $u$ -channel, where the exchanged states may again belong to a different sector. For example, the four-point function (3.39) is  $s \leftrightarrow t$  symmetric, with  $L_0^{\text{eff}} = \frac{1}{2}$  states exchanged in both these channels; if expanded instead in the  $u$ -channel, the exchanged states have  $L_0^{\text{eff}} = 1$ . The other four-point functions in section 3 provide further examples.

#### 4.3 The relation with the Ising model

Indications of a relation with the Ising model have been apparent since the earliest work on the  $W_3$  string [1,2,5]. Until now, the evidence was essentially numerological, consisting of a twofold observation. Firstly, the central charge of the effective energy-momentum tensor  $T^{\text{eff}}$  given in (2.5) can be written as  $\frac{51}{2} = 26 - \frac{1}{2}$ , where 26 is the critical central charge of the usual bosonic string and  $\frac{1}{2}$  is the central charge of the Ising model. Secondly, the set of  $L_0^{\text{eff}}$  values, namely  $\{1, \frac{15}{16}, \frac{1}{2}\}$ , can be written as  $1 - \Delta_{\text{min}}$ , where 1 is the intercept of the usual bosonic string and  $\Delta_{\text{min}}$  takes the values of the dimensions of the primary fields of the Ising model, namely  $\{0, \frac{1}{16}, \frac{1}{2}\}$ .

Inspired by this numerological connection, it was recently proposed in [18] that one might be able to compute the scattering amplitudes for those physical states of the  $W_3$  string that admit an effective spacetime interpretation by tensoring, by hand, the effective spacetime parts of the physical states with appropriate primary fields of the Ising model, and then calculating the scattering amplitudes for the tensor-product states. This procedure implicitly assumes that the physical operators of the  $W_3$  string are equivalent to direct products of effective Virasoro operators with Ising fields. It is *a priori* far from clear that this direct product structure captures the essence of the  $W_3$  symmetry. Later in this subsection we shall examine this assumption in more detail in the light of our calculation of the  $W_3$ -string scattering amplitudes, and show that indeed there is more to the  $W_3$  symmetry than can be described by a direct-product structure.

It follows from our results in section 3 that the connection between the  $W_3$  string and the Ising model is more than numerological. In fact, the pattern of vanishing and non-vanishing three-point functions computed in subsection 3.3 reproduces the fusion rules of the Ising model. To see this, we associate, as suggested by the numerological observation, the  $L_0^{\text{eff}} = 1$  sector with the identity operator  $\mathbf{1}$  of the Ising model; the  $L_0^{\text{eff}} = \frac{15}{16}$  sector with the spin operator  $\sigma$ ; and the  $L_0^{\text{eff}} = \frac{1}{2}$  sector with the energy operator  $\varepsilon$ . It is now immediately clear that the structure of the three-point functions presented in subsection 3.3 exactly agrees with the fusion rules [19] of the Ising model, *viz.*

$$\begin{aligned} \mathbf{1} \times \mathbf{1} &= \mathbf{1} , & \sigma \times \sigma &= \mathbf{1} + \varepsilon , \\ \mathbf{1} \times \sigma &= \sigma , & \sigma \times \varepsilon &= \sigma , \\ \mathbf{1} \times \varepsilon &= \varepsilon , & \varepsilon \times \varepsilon &= \mathbf{1} . \end{aligned} \tag{4.3}$$

As we already mentioned, the three-point function describes the basic interaction vertex in the  $W_3$  string. Thus one may expect, and indeed it is the case, that all the four-point functions that we have computed should be consistent with the above fusion rules and the corresponding four-point functions of the Ising model, in the sense that an amplitude that is forbidden by the fusion rules (4.3) is indeed zero in the  $W_3$  string. In particular, this explains the vanishing of (3.43) and (3.44). However, it appears that this correspondence does not go both ways. From the fusion rules of the Ising model, one might expect that there should exist four-point functions for four  $L_0^{\text{eff}} = 1$  operators and also for four  $L_0^{\text{eff}} = \frac{15}{16}$  operators. With the physical operators given in this paper, we cannot construct either of these; in neither case it is possible to have the correct total ghost structure together with momentum conservation in the  $\varphi$  direction. For the  $L_0^{\text{eff}} = 1$  case, this may simply be because we have not explored higher-level states in the physical spectrum sufficiently. However, for the  $L_0^{\text{eff}} = \frac{15}{16}$  case on the other hand, it seems impossible, from the general pattern of physical states that is known so far, to construct such a four-point function satisfying  $\beta$ -momentum conservation. To see this, suppose that, as it is the case for all known physical states, the  $\beta$  momentum is of the form  $\beta = \frac{k}{7}Q$ , with  $k$  an integer.\* It follows that a physical state with  $L_0^{\text{eff}} = \frac{15}{16}$  would have  $k = -7 \pm 4\sqrt{n}$ , where  $n$  is an integer related to the level number. To satisfy momentum conservation in the four-point function, we would need four such (integer)  $k$ 's satisfying  $k_1 + k_2 + k_3 + k_4 = -14$ , which is manifestly impossible. (In fact even if one relaxes the supposition that the  $k$ 's are integers, it would still be the case that momentum conservation could not be satisfied for such a four-point function. This follows from the relatively-easily proven fact that, for integer  $n_i$ , the sum  $\pm\sqrt{n_1} \pm \sqrt{n_2} \pm \sqrt{n_3} \pm \sqrt{n_4}$  cannot equal  $\frac{7}{2}$ .)

Let us now compare this result with the computation in [18] of the four-point function for four  $L_0^{\text{eff}} = \frac{15}{16}$  effective Virasoro operators tensored with four spin- $\frac{1}{16}$  fields of the Ising model. Interestingly, this computation gave a non-zero result. In view of our finding above, namely that the four-point function of four  $L_0^{\text{eff}} = \frac{15}{16}$  physical operators in the  $W_3$  string is zero, it seems that essential aspects of the  $W_3$  symmetry are not captured by the method of [18]. It appears from our results that a  $W_3$ -symmetry selection rule forbids the existence of such a non-vanishing four-point function in the  $W_3$  string.

We have checked that if one uses the method in [18] to calculate the four-point function for four  $L_0^{\text{eff}} = \frac{1}{2}$  effective Virasoro operators tensored with four spin- $\frac{1}{2}$  Ising fields, the result is the same as the result (3.30) for the scattering of four  $L_0^{\text{eff}} = \frac{1}{2}$  physical states of the  $W_3$  string. It is not clear to us to what extent in general the method presented in [18] should agree with the  $W_3$ -string scattering amplitudes, which are described in this paper.

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\* In fact the quantisation of  $\beta$  in units of  $\frac{1}{7}Q$  follows from a functional integral approach if  $\varphi$  is taken as a time-like coordinate, since under this circumstance  $\varphi$  is automatically periodic [4,5].

## 5. Open problems and conclusions

In this paper we have presented a procedure for computing gauge-invariant scattering amplitudes in the  $W_3$  string. Although we have concentrated on the open string, the procedure is equally applicable to the closed string. The essential point that enables us to build scattering amplitudes is the existence of physical states with non-standard ghost structure; their inclusion is vital, since there seems to be no way to obtain non-vanishing scattering amplitudes amongst physical states with only standard ghost structure.

All the physical states we have considered in this paper have the property of factorising into a product of the form

$$|\text{effective spacetime}\rangle \otimes |\text{ghost} + \varphi\rangle . \quad (5.1)$$

We have observed in several examples that the  $L_0^{\text{eff}}$  value characterises these physical states, in the sense that states with the same  $L_0^{\text{eff}}$  but different ghost structure and  $\varphi$  dependence behave equivalently in correlation functions.

For simplicity, we have restricted our attention to physical states of the form (5.1) where  $|\text{effective spacetime}\rangle$  is a tachyonic state. The only property of  $|\text{effective spacetime}\rangle$  that is relevant when imposing the physical state conditions on a state of the form (5.1) is that it should be a highest-weight state under  $T^{\text{eff}}$  with weight  $L_0^{\text{eff}}$ . This means that we can replace the effective tachyonic state by an arbitrary excited effective physical state with the same intercept  $L_0^{\text{eff}}$ . One can then straightforwardly write down scattering amplitudes for these new excited physical states.

There are, however, many states in the physical spectrum of the  $W_3$  string that are not of the form (5.1). Specifically, they have prefactors that are the sum of terms that involve  $X^\mu$  excitations and terms that do not, and thus they do not factorise into the form (5.1). The procedure that we have developed in this paper for calculating scattering amplitudes is equally applicable for such states. The effective spacetime interpretation of these physical states, and of the corresponding scattering amplitudes, is not yet clear. In addition, it is not clear how, if at all, these non-factorisable states are related to the Ising model.

An important issue that has not been addressed in this paper is the question of unitarity. For physical states with standard ghost structure, unitarity was proven in [5], by exploiting the fact that they all admit an effective spacetime interpretation with effective intercept values  $L_0^{\text{eff}} = \frac{15}{16}$  or  $L_0^{\text{eff}} = 1$ . One can easily show that the unitarity bounds for an effective Virasoro string with central charge  $c = \frac{51}{2}$  imply that the effective intercept must satisfy either  $\frac{15}{16} \leq L_0^{\text{eff}} \leq 1$  or  $0 \leq L_0^{\text{eff}} \leq \frac{1}{2}$  [5]. Thus the standard-type physical states of the  $W_3$  string precisely saturate the lower and upper limits of the first of these unitarity bounds. In fact *all* the known physical states of the  $W_3$  string that admit an effective spacetime interpretation saturate one or other of the unitarity bounds, since they have  $L_0^{\text{eff}} = 1, \frac{15}{16}$  or  $\frac{1}{2}$ . Further work on the question of unitarity for the  $W_3$  string is in progress.

We restricted attention in this paper to physical operators at ghost number  $G = 3$  (for operators with standard ghost structure), or  $G = 2$  (for operators with non-standard ghost structure), and their boosted partners at  $G = 4$  or  $G = 3$  respectively. It is easy to see that when  $N$  is greater than 5,  $N$ -point functions of such operators can never have the correct ghost structure, and thus they all vanish. However, as we mentioned in section 2, there are also physical operators with ghost number  $G = 0$ ; the first non-trivial example (*i.e.* with *continuous* on-shell spacetime momentum  $p_\mu$ ) occurs at level  $\ell = 8$ , with  $\beta = \frac{4}{7}Q$  [16]. Further examples will arise at higher levels, and can be generated by the action of the ground-ring generators of the  $W_3$  string [10,16]. With such states the  $N \leq 5$  limit discussed above can clearly be overcome. It is interesting to note also that these  $G = 0$  operators have positive  $\beta$  momentum, and can thus counterbalance the negative contributions from the  $G = 2$  and  $G = 3$  operators discussed in this paper.

The procedure that we have developed in this paper seems to provide a consistent picture of  $W_3$ -string scattering. Although we can calculate any desired scattering amplitude, the underlying structure remains obscure. This is not surprising, since elucidating the organising principle would require the understanding of  $W$  geometry. It may be, however, that our results provide us with a glimpse of  $W$  geometry itself. By analogy with the super-extension of the bosonic string, where one introduces superspace and integrates over it, one should expect that in a  $W$  extension of the bosonic string one should introduce a “ $W$  space” and “integrate” over it. In view of our results it is not inconceivable that this integration over  $W$  space turns states with standard ghost structure into states with non-standard ghost structure and *vice versa*. This suggests that  $W$  geometry cannot be completely understood without including ghosts.

In a usual gauge theory the rôle of the ghost fields is to remove the unphysical degrees of freedom of the gauge fields. As such, they appear only as virtual particles, and never as external states in physical amplitudes. In the  $W_3$  string, on the other hand, their rôle seems to be strikingly different: ghost excitations must necessarily appear in the external states of physical amplitudes. Whereas ghost fields in external states would spoil unitarity in a usual gauge theory, such as Yang-Mills, it seems plausible that they are *needed* for unitarity of the  $W_3$  string. Indeed the  $W_3$  string differs in an essential way from a usual gauge theory in that the gauge algebra of the matter fields in the quantum theory, namely  $W_3$ , is different from the gauge algebra of the original classical theory, which is a contraction of  $W_3$ . This non-trivial renormalisation of the gauge algebra does not happen in a usual gauge theory. It may well be, therefore, that it is inappropriate to try to understand  $W_3$ -string theory from a classical point of view.

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## APPENDIX

In this appendix we present the explicit forms of the various physical operators that we use in this paper. In particular, we give the operators corresponding to all the prime states at level 0 (*i.e.* with standard ghost structure), level 1 and level 2, and one example at level 3 (these all have non-standard ghost structure). We recall that we denote such an operator by  $\mathbf{V}_\Delta^G[\beta, p]$ , where  $G$  is its ghost number,  $\Delta$  is its  $L_0^{\text{eff}}$  value, and  $\beta$  and  $p$  are the momenta in the  $\varphi$  and effective spacetime ( $X^\mu$ ) directions. For level 0 and level 1, we also give for each prime state a member of the multiplet whose ghost number has been boosted by 1. Such operators are denoted by  $\mathbf{W}_\Delta^G[\beta, p]$ . In each case we choose a special linear combination of the ghost boosters  $a_\varphi$  and  $a_{X^\mu}$  with which we act on the prime state. This combination is chosen so as to introduce no excitations in the  $X^\mu$  directions, in order to preserve the factorisability (5.1) and consequently to permit the same effective spacetime interpretation for the boosted physical states that the prime states enjoy. Expressions for  $a_\varphi$  and  $a_{X^\mu}$  are presented in [10] for the two-scalar case, and can be generalised immediately to the multi-scalar case.

- *Standard ghost structure: level 0*

$$\mathbf{V}_1^3[-\frac{8}{7}Q, p] = c \partial \gamma \gamma e^{-\frac{8}{7}Q\varphi} e^{ip \cdot X}, \quad (A.1)$$

$$\mathbf{V}_1^3[-\frac{6}{7}Q, p] = c \partial \gamma \gamma e^{-\frac{6}{7}Q\varphi} e^{ip \cdot X}, \quad (A.2)$$

$$\mathbf{V}_{15/16}^3[-Q, p] = c \partial \gamma \gamma e^{-Q\varphi} e^{ip \cdot X}. \quad (A.3)$$

- *Boosted operators*

$$\mathbf{W}_1^4[-\frac{8}{7}Q, p] = c \partial^2 \gamma \partial \gamma \gamma e^{-\frac{8}{7}Q\varphi} e^{ip \cdot X}, \quad (A.4)$$

$$\mathbf{W}_1^4[-\frac{6}{7}Q, p] = c \partial^2 \gamma \partial \gamma \gamma e^{-\frac{6}{7}Q\varphi} e^{ip \cdot X}, \quad (A.5)$$

$$\mathbf{W}_{15/16}^4[-Q, p] = c \partial^2 \gamma \partial \gamma \gamma e^{-Q\varphi} e^{ip \cdot X}. \quad (A.6)$$



- *Non-standard ghost structure: level 1*

$$\mathbf{V}_{15/16}^2[-\frac{3}{7}Q, p] = \left(c\gamma + \frac{i}{3\sqrt{58}}\partial\gamma\gamma\right)e^{-\frac{3}{7}Q\varphi}e^{ip\cdot X}, \quad (A.7)$$

$$\mathbf{V}_{1/2}^2[-\frac{4}{7}Q, p] = \left(c\gamma - \frac{i}{3\sqrt{58}}\partial\gamma\gamma\right)e^{-\frac{4}{7}Q\varphi}e^{ip\cdot X}. \quad (A.8)$$

- *Boosted operators*

$$\mathbf{W}_{15/16}^3[-\frac{3}{7}Q, p] = \left(11c\partial^2\gamma\gamma - 12\sqrt{2}\partial\varphi c\partial\gamma\gamma - \frac{13i}{\sqrt{58}}\partial^2\gamma\partial\gamma\gamma\right)e^{-\frac{3}{7}Q\varphi}e^{ip\cdot X}, \quad (A.9)$$

$$\mathbf{W}_{1/2}^3[-\frac{4}{7}Q, p] = \left(10c\partial^2\gamma\gamma - 12\sqrt{2}\partial\varphi c\partial\gamma\gamma - \frac{38i}{3\sqrt{58}}\partial^2\gamma\partial\gamma\gamma\right)e^{-\frac{4}{7}Q\varphi}e^{ip\cdot X}. \quad (A.10)$$

- *Non-standard ghost structure: level 2*

$$\begin{aligned} \mathbf{V}_{1/2}^2[-\frac{2}{7}Q, p] = & \frac{-i}{\sqrt{29}}\left(\partial\varphi\partial\gamma\gamma + \sqrt{58}i\partial\varphi c\gamma - \frac{3}{2}\sqrt{29}ic\partial\gamma\right. \\ & \left.- \frac{2}{3}\sqrt{2}\partial^2\gamma\gamma - \frac{1}{3}\sqrt{2}bc\partial\gamma\gamma\right)e^{-\frac{2}{7}Q\varphi}e^{ip\cdot X}. \end{aligned} \quad (A.11)$$

- *Non-standard ghost structure: level 3*

$$\begin{aligned} \mathbf{V}_1^2[0, p] = & \left(\frac{680}{261}b\partial^2\gamma\partial\gamma\gamma - 36c\beta\partial\gamma\gamma - 19c\partial^2\gamma - \frac{16}{29}\sqrt{29}i\partial\varphi bc\partial\gamma\gamma + 42\sqrt{2}\partial\varphi c\partial\gamma\right. \\ & - 24(\partial\varphi)^2c\gamma + \frac{28}{29}\sqrt{58}i(\partial\varphi)^2\partial\gamma\gamma - \frac{172}{87}\sqrt{29}i\partial\varphi\partial^2\gamma\gamma - \frac{14}{29}\sqrt{58}i\partial bc\partial\gamma\gamma \\ & \left.+ \frac{48}{29}\sqrt{29}i\partial^2\varphi\partial\gamma\gamma + \frac{301}{174}\sqrt{58}i\partial^2\gamma\partial\gamma\right)e^{ip\cdot X}. \end{aligned} \quad (A.12)$$

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